there is a supporting hyperplane H for E through e. A vector p orthogonal to the hyperplane H, pointing towards E has all the required properties. The treatment of the problem thus given by means of convexity theory was rigorous, more general and simpler than the treatment by means of the differential calculus that had been traditional since Pareto. The supporting hyperplane theorem (more generally the Hahn-Banach theorem, Debreu [1954]) seemed to lit the economic problem perfectly. Especially relevant to my narrative is the fact that the restatement of welfare economics in set-theoretical terms forced a reexamination of several of the primitive concepts of the theory of general economic equilibrium. This was of great value for the solution of the existence problem.

In the year I joined the Cowles Commission, I learned about the Lemma in von Neumann's article of 1937 on growth theory that Shizuo Kakutani reformulated in 1941 as a fixed point theorem. I also learned about the applications of Kakutani's theorem made by John Nash in his one-page note of 1950 on "Equilibrium Points in N-Person Games" and by Morton Slater in his unpublished paper, also of 1950, on Lagrange multipliers. Again there was an ideal tool, this time Kakutani's theorem, for the proof that I gave in 1952 of the existence of a social equilibrium generalizing Nash's result. Since the transposition from the case of two agents to the case of n agents is immediate, we shall consider only the former which lends itself to a diagrammatic representation. Let the first agent choose an action $a_1$ in the ***a priori*** given set $A_1$, and the second agent choose an action $a_2$ in the ***a priori*** given set $A_2$. Knowing $a_2$, the first agent has a set $\mu_1(a_2)$ of equivalent reactions. Similarly, knowing $a_1$, the second agent has a set $\mu_2(a_1)$ of equivalent reactions. $\mu_1(a_2)$ and $\mu_2(a_1)$ may be one-element

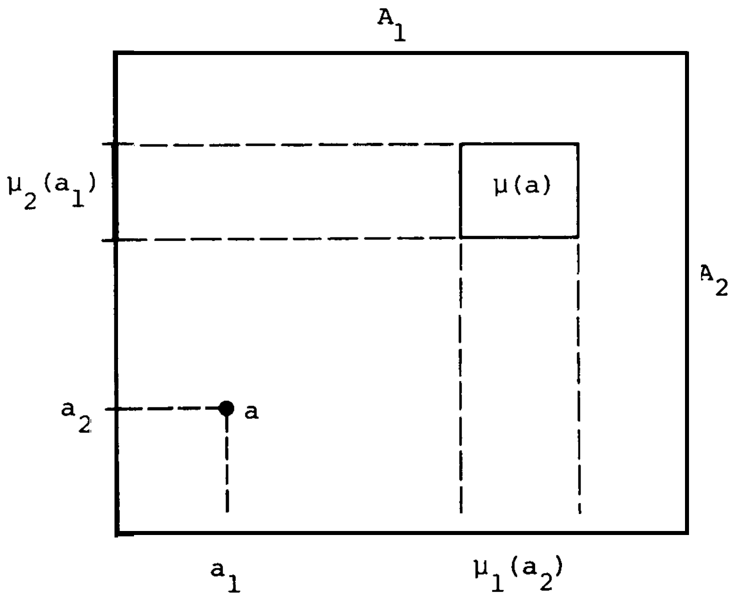


*Figure 2*

sets, but in the important case of an economy with some producers operating under constant returns to scale, they will not be. The state $a = (a_1,a_2)$ is an equilibrium if and only if $a_1 \in \mu_1(a_2)$ and $a_2 \in \mu_2(a_1)$, that is if and only if $a \in \mu(a) = \mu_1(a_2) \times \mu_2(a_1)$.

In other words, a is an equilibrium state if and only if it is a fixed point of the correspondence $a \mapsto \mu(a)$ from $A = A_1 x A_2$ to A itself. Conditions insuring that Kakutani's theorem applies to A and μ guarantee the existence of an equilibrium state. In our article of 1954, Arrow and I cast a competitive economy in the form of a social system of the preceding type. The agents are the consumers, the producers, and a fictitious price-setter. An appropriate definition of the set of reactions of the price-setter to an excess demand vector makes the concept of equilibrium for that social system equivalent to the concept of competitive equilibrium for the original economy. In this manner a proof of existence, resting ultimately on Kakutani's theorem, was obtained for an equilibrium of an economy made up of interacting consumers and producers. In the early fifties, the time had undoubtedly come for solutions of the existence problem. In addition to the work of Arrow and me, begun independently and completed jointly, Lionel McKenzie at Duke University proved the existence of an "Equilibrium in Graham's Model of World Trade and Other Competitive Systems" [1954], also using Kakutani's theorem. A different approach taken independently by David Gale [1955] in Copenhagen, Hukukane Nikaido [1956] in Tokyo, and Debreu [1956] in Chicago permitted the substantial simplification given in my *Theory of Value* [1959] of the complex proof of Arrow and Debreu.

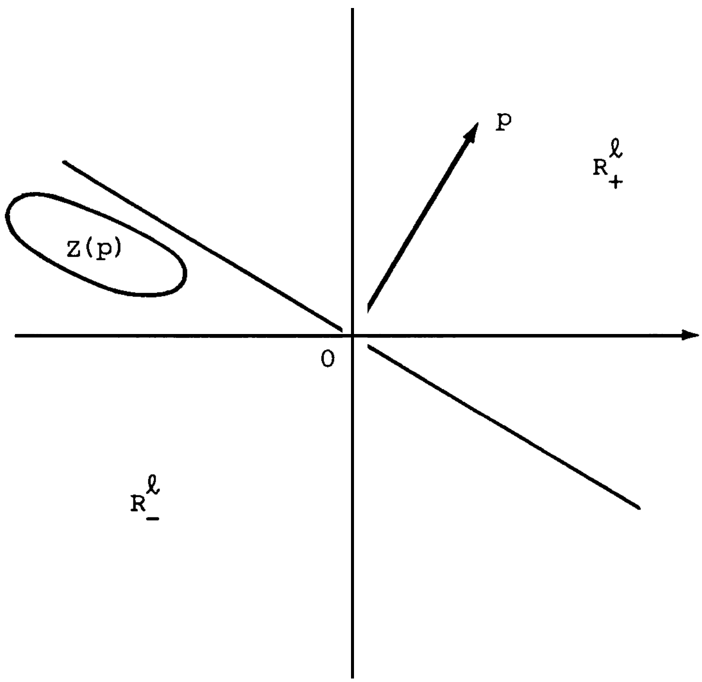


*Figure 3*

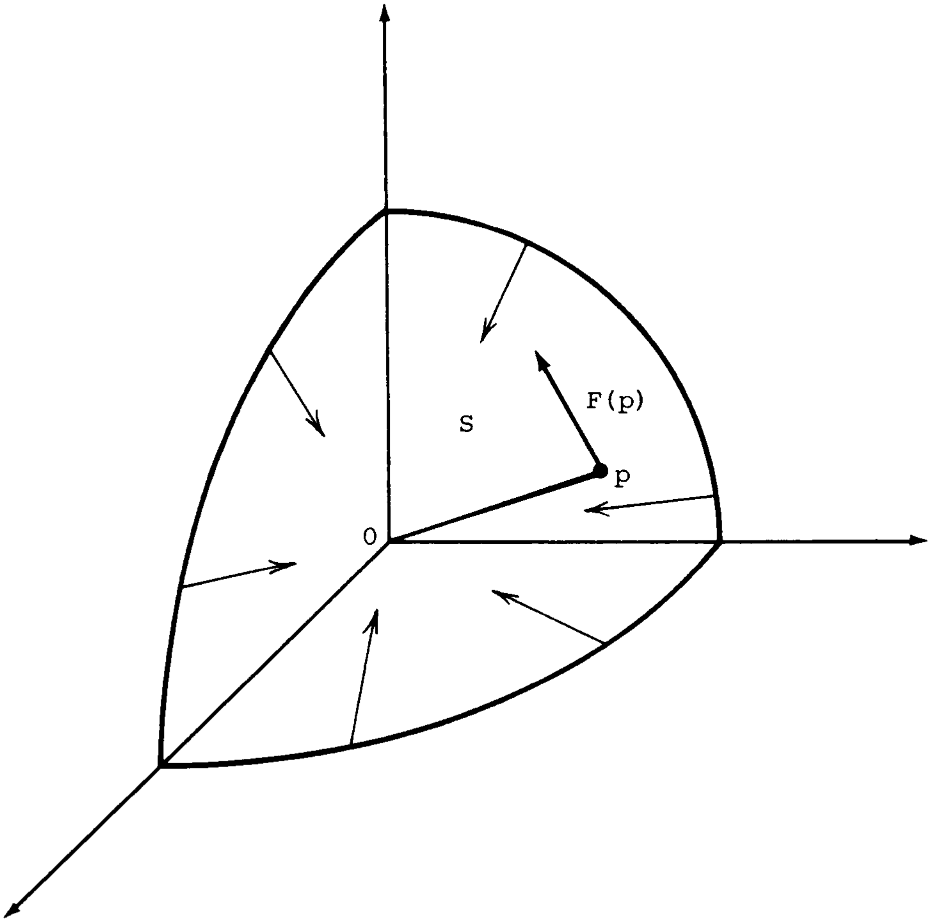


*Figure 4*

natural part of the program of study of general economic equilibrium. Yet the decisive stimulus came unexpectedly from the solution of a problem in game theory, when Lemke and Howson [1964] provided an algorithm for the solution of two-person non-zero-sum games. The computation of equilibria has found its way into a large number of applications and has added an important new aspect to the theory of general economic equilibrium.

The explanation of equilibrium given by a model of the economy would be complete if the equilibrium were unique, and the search for satisfactory conditions guaranteeing uniqueness has been actively pursued (an excellent survey is found in Arrow and Hahn [1971], Chapter 9). However, the strength of the conditions that were proposed made it clear by the late sixties that global uniqueness was too demanding a requirement and that one would have to be satisfied with local uniqueness. Actually, that property of an economy could not be guaranteed even under strong assumptions about the characteristics of the economic agents. But one can prove, as I did in 1970, that, under suitable conditions, in the set of all economies, the set of economies that do ***not*** have a set of locally unique equilibria is negligible. The exact meaning of the terms I have

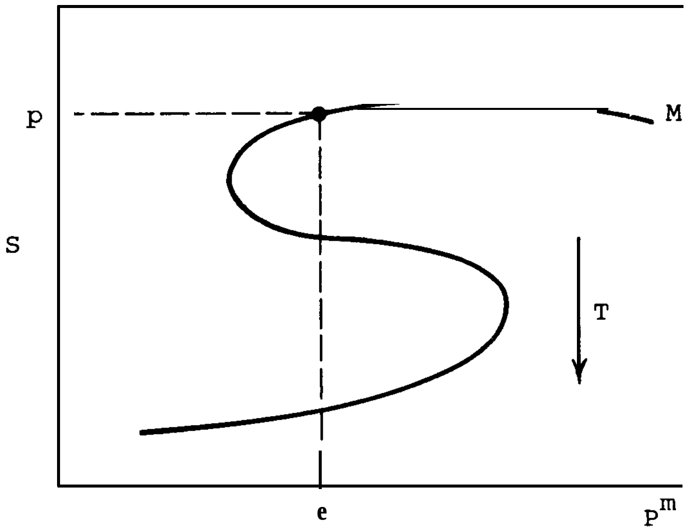


*Figure 5*

Smale, Yves Balasko, and Andreu Mas-Colell [1984] are among its main contributors.

Departing from chronological order, I now return to the late fifties and to the early sixties, and to the beginning of the theory of the core of an economy. Edgeworth [1881] had given a persuasive argument in support of the common imprecise belief that markets become more competitive as the number of their agents increases in such a way that each one of them tends to become negligible. He had specifically shown that his "contract-curve" tends to the set of competitive equilibria in a two-commodity economy with equal numbers of consumers of each one of two types. His brilliant contribution stimulated no further work until Martin Shubik [1959] linked Edgeworth's contract curve with the game theoretical concept of the core (Gillies [1953]). The first extension of Edgeworth's result was obtained by Herbert Scarf [1962], and the complete generalization to the case of an arbitrary number of commodities and of types of consumers was given by Debreu and Scarf [1963]. Associated with our joint paper is one of my most vivid memories of the instant when a problem is solved. Herbert Scarf, then at Stanford, had met me at the San Francisco airport in December, 1961, and as he was driving to Palo Alto on the freeway, one of us, in one sentence, provided a key to the solution; the other, also in one sentence, immediately provided the other key; and the lock clicked open. Once again, the basic mathematical result was the supporting hyperplane theorem for convex sets. The theorem that we had proved remained special, because it applied only to economies with a given number of types of consumers and an equal, increasing number of consumers of each type. Generalizations were soon forthcoming. Robert Aumann [1964] introduced the concept of an atomless measure space of economic agents, a natural mathematical formulation of the concept of an economy with a large number of agents, all of them negligible. Under notably weak conditions, Aumann proved that for such an economy the

from which exploration could start in new directions. It has freed researchers from the necessity of questioning the work of their predecessors in every detail. Rigor undoubtedly fulfills an intellectual need of many contemporary economic theorists, who therefore seek it for its own sake, but it is also an attribute of a theory that is an effective thinking tool. Two other major attributes of an effective theory are simplicity and generality. Again, their aesthetic appeal suffices to make them desirable ends in themselves for the designer of a theory. But their value to the scientific community goes far beyond aesthetics. Simplicity makes a theory usable by a great number of research workers. Generality makes it applicable to a broad class of problems.

In yet another manner, the axiomatization of economic theory has helped its practitioners by making available to them the superbly efficient language of mathematics. It has permitted them to communicate with each other, and to think, with a great economy of means. At the same time, the dialogue between economists and mathematicians has become more intense. The example of a mathematician of the first magnitude like John von Neumann devoting a significant fraction of his research to economic problems has not been unique. Simultaneously, economic theory has begun to influence mathematics. Among the clearest instances are Kakutani's theorem, the theory of integration of correspondences (Hildenbrand [1974]), algorithms for the computation of approximate fixed points (Scarf's Chapter 21 in Arrow and Intriligator [1981-4]), and of approximate solutions of systems of equations (Smale's Chapter 8 in Arrow and Intriligator [1981-4]).

## III.

In narratives of their careers, scientists try to acknowledge the main influences to which they responded, and the support they received from other scientists and from different institutions, even though such attempts are unlikely to be entirely successful. To all the persons and organizations I have named, I want to add the outstanding education system I have known in France, and the Centre National de la Recherche Scientifique which made my conversion from mathematics to economics possible. After my move to the United States in 1950, I was associated with three great universities (Chicago, Yale, and Berkeley) where scientific research is a natural way of life; and during the last two decades the Economics Program of the National Science Foundation has given me, more than anything else, time for that research. All those institutions have provided a superb environment for the task that had to be performed.